

A FORMULA FOR THE HYPERGEOMETRIC FUNCTION OF TYPE BC_n

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ABSTRACT. Formulae of Berezin and Karpelevič for the radial parts of invariant differential operators and the spherical function on a complex Grassmann manifold are generalized to the hypergeometric functions associated with root system of type BC_n under condition that the multiplicity of the middle roots is zero or one.

INTRODUCTION

Berezin and Karpelevič [3] gave an explicit expression for radial parts of invariant differential operators and the spherical functions on $SU(p, q)/S(U(p) \times U(q))$ without proof and Hoogenboom [7] gave proof of these results. Explicit expressions of the Laplace-Beltrami operator and higher order invariant differential operators allows us to construct eigenfunctions by the method of separation of variables and the spherical function has an expression using determinant of a matrix whose entries are the Gauss hypergeometric functions.

Heckman and Opdam developed theory of the hypergeometric function associated with a root system, which is a generalization of theory of the spherical function on a symmetric space ([5]). Namely, the radial part of the Laplace-Beltrami operator of a Riemannian symmetric space of the non-compact type consists of data such as the restricted root system, multiplicities of roots. Heckman and Opdam allowed multiplicities of roots arbitrary complex numbers (that coincide on every Weyl group orbit) and constructed commuting family of differential operators and eigenfunctions. For rank one (one variable) case, their hypergeometric function is the Jacobi function ([8]), which is essentially the same as the Gauss hypergeometric function.

In this paper, the author proves that the results of Berezin and Karpelevič [3] are valid for the hypergeometric function associated with root system of type BC_n under the condition that the multiplicity of the middle roots is 1. Though it is an easy generalization of [3], our results cover integral middle multiplicities in conjunction with the hypergeometric shift operator, which include many cases of symmetric spaces.

1. HYPERGEOMETRIC FUNCTION ASSOCIATED WITH A ROOT SYSTEM

1.1. Notation. In this section, we review on the hypergeometric function associated with a root system. See [5] for details.

Let E be an n -dimensional Euclidean space with inner product (\cdot, \cdot) . For $\alpha \in E$ with $\alpha \neq 0$ write

$$(1.1) \quad \alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$

Let $R \subset E$ be a root system of rank n and W its Weyl group. Let $R_+ \subset R$ be a fixed set of positive roots and $E_+ \subset E$ be the corresponding positive Weyl chamber. Let

$$(1.2) \quad P = \{\lambda \in E : (\lambda, \alpha^\vee) \in \mathbb{Z} \ \forall \alpha \in R\}.$$

Let $k_\alpha (\alpha \in R)$ be complex numbers such that $k_{w\alpha} = k_\alpha$ for all $w \in W$. We call $k = (k_\alpha)_{\alpha \in R}$ a multiplicity function on R . Let K denote the set of multiplicity function on R . We put

$$(1.3) \quad \rho(k) = \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha,$$

$$(1.4) \quad \delta(k) = \prod_{\alpha \in R_+} (e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha})^{2k_\alpha}.$$

1.2. Commuting family of differential operators. Let ξ_1, \dots, ξ_n be an orthonormal basis of E and consider the differential operator

$$(1.5) \quad L(k) = \sum_{j=1}^n \partial_{\xi_j}^2 + \sum_{\alpha \in R_+} k_\alpha \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \partial_\alpha$$

on E . Here ∂_α denotes the directional derivative along α such that $\partial_\alpha(e^\lambda) = (\alpha, \lambda)e^\lambda$ for $\alpha, \lambda \in E$. We have

$$(1.6) \quad \begin{aligned} & \delta(k)^{\frac{1}{2}} \circ \{L(k) + (\rho(k), \rho(k))\} \circ \delta(k)^{-\frac{1}{2}} \\ &= \sum_{j=1}^n \partial_{\xi_j}^2 + \sum_{\alpha \in R_+} \frac{k_\alpha(1 - k_\alpha - 2k_{2\alpha})(\alpha, \alpha)}{(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha})^2}. \end{aligned}$$

Let \mathcal{R} denote the algebra generated by the functions

$$(1.7) \quad \frac{1}{1 - e^{-\alpha}} \quad (\alpha \in R_+)$$

viewed as a subalgebra of the quotient field of $\mathbb{R}[P]$. Let $S(E)$ denote the symmetric algebra of E . Let $\mathbb{D}_{\mathcal{R}} = \mathcal{R} \otimes S(E)$ denote the algebra of differential operators on E with coefficient in \mathcal{R} and let $\mathbb{D}_{\mathcal{R}}^W$ be the subalgebra of W -invariants in $\mathbb{D}_{\mathcal{R}}$. Let $\gamma(k)$ denote the algebra homomorphism

$$(1.8) \quad \gamma(k) : \mathbb{D}_{\mathcal{R}} \longrightarrow S(E)$$

defined by

$$\gamma(k) \left(\frac{1}{1 - e^{-\alpha}} \right) = 1 \quad (\alpha \in R_+).$$

Let

$$(1.9) \quad \mathbb{D}(k) = \{D \in \mathbb{D}_{\mathcal{R}}^W : [L(k), P] = 0\}$$

denote the commutator of $L(k)$ in $\mathbb{D}_{\mathcal{R}}^W$ and let $S(E)^W$ denote the set of W -invariants in $S(E)$.

Theorem 1.1. *The map*

$$(1.10) \quad \gamma(k) : \mathbb{D}(k) \longrightarrow S(E)^W$$

is an algebra isomorphism. In particular, $\mathbb{D}(k)$ is a commutative algebra. Moreover, if $D \in \mathbb{D}_{\mathcal{R}}^W$ is a differential operator of order N , then its principal symbol $\sigma(D)$ has constant coefficients and coincides with homogeneous component of $\gamma(k)(D)$ of degree N .

1.3. The hypergeometric function. Let Q be the root lattice

$$(1.11) \quad Q = \{\sum_{\alpha \in R_+} z_{\alpha} \alpha : z_{\alpha} \in \mathbb{Z}_+\}.$$

Put

$$(1.12) \quad \mathfrak{h} = E_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} E, \quad A = \exp E, \quad e = \exp 0, \quad A_+ = \exp E_+$$

For $\mu \in \mathfrak{h}^*$ and $a \in A$, we write $a^{\mu} = \exp(\mu(\log a))$.

If $\lambda \in \mathfrak{h}^*$ satisfies the condition

$$(1.13) \quad -2(\lambda, \mu) + (\mu, \mu) \neq 0 \text{ for all } \mu \in Q,$$

then the equation

$$(1.14) \quad L(k)u = ((\lambda, \lambda) - (\rho(k), \rho(k))u$$

has a unique solution on A_+ of the form

$$(1.15) \quad u(a) = \Phi(\lambda, k; a) = \sum_{\mu \in Q} \Gamma_{\mu} a^{\lambda - \rho(k) - \mu}$$

with $\Gamma_0 = 1$. The function $\Phi(\lambda, k; a)$ is also a solution of the system of differential equations

$$(1.16) \quad Du = \gamma(k)(D)(\lambda)u, \quad D \in \mathbb{D}(k).$$

If

$$(\lambda, \alpha^{\vee}) \notin \mathbb{Z} \text{ for all } \alpha \in R,$$

then $\Phi(w\lambda, k; a)$ ($w \in W$) form a basis of the solution space of (1.16).

Define meromorphic functions \tilde{c} and c on $\mathfrak{h} \times K$ by

$$(1.17) \quad \tilde{c}(\lambda, k) = \prod_{\alpha \in R_+} \frac{\Gamma((\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{1}{2}\alpha})}{\Gamma((\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{1}{2}\alpha} + k_\alpha)}$$

and

$$(1.18) \quad c(\lambda, k) = \frac{\tilde{c}(\lambda, k)}{\tilde{c}(\rho(k), k)}$$

with the convention $k_{\frac{1}{2}\alpha} \neq 0$ if $\frac{1}{2}\alpha \notin R$. We call the function

$$(1.19) \quad F(\lambda, k; a) = \sum_{w \in W} c(w\lambda, k) \Phi(w\lambda, k; a)$$

the hypergeometric function associated with R . Let $S \subset K$ denote the set of zeroes of $\tilde{c}(\rho(k), k)$.

Theorem 1.2. *Assume that $k \in K \setminus S$. Then the system of differential equation (1.16) has a unique solution that is regular at $e \in A$, W -invariant, and*

$$F(\lambda, k; e) = 1.$$

The function F is holomorphic in $\lambda \in \mathfrak{h}$, $k \in K \setminus S$, and analytic in $a \in A$.

Remark 1.3. Theorem 1.1 and Theorem 1.2 were proved by Heckman and Opdam in a series of papers. See [5] and references therein.

Let G/K be a Riemannian symmetric space of the non-compact type, Σ be the restricted root system, and m_α be the root multiplicity (dimension of the root space) of $\alpha \in \Sigma$. Put

$$(1.20) \quad R = 2\Sigma, \quad k_{2\alpha} = \frac{1}{2}m_\alpha.$$

Then (1.5) is the radial part of the Laplace-Beltrami operator on G/K , $\mathbb{D}(k)$ is the algebra of radial parts of invariant differential operators on G/K , and $F(\lambda, k; a)$ is the radial part of the spherical function on G/K . In this case Theorem 1.1 and Theorem 1.2 were previously proved by Harish-Chandra. See [6] for theory of spherical functions on symmetric spaces.

1.4. Rank one case. For a root system of rank 1, the hypergeometric function is given by the Jacobi function. We review on the Jacobi function. See [8] for details.

Assume that $R = \{\pm e_1, \pm 2e_1\}$ with $(e_1, e_1) = 1$ and put

$$(1.21) \quad k_s = k_{e_1}, \quad k_l = k_{2e_1}, \quad \alpha = k_s + k_l - 1/2, \quad \beta = k_l - 1/2.$$

We identify $\lambda \in \mathfrak{a}_\mathbb{C}^*$ with $(\lambda, 2e_1) \in \mathbb{C}$ and let $t = e_1(\log a)/2$ be a coordinate on $A \simeq \mathbb{R}$. Then

$$(1.22) \quad \rho(k) = k_s + 2k_l = \alpha + \beta + 1.$$

The hypergeometric system (1.16) turns out to be the differential equation

$$(1.23) \quad L(k)F = (\lambda^2 - \rho(k)^2)F,$$

where

$$(1.24) \quad L(k) = \frac{d^2}{dt^2} + 2(k_s \coth t + 2k_l \coth 2t) \frac{d}{dt}$$

and the hypergeometric function $F(\lambda, k; a_t)$ of type BC_1 is given by the Jacobi function

$$(1.25) \quad F(\lambda, k; a_t) = \varphi_{\sqrt{-1}\lambda}^{(\alpha, \beta)}(t) = {}_2F_1\left(\frac{1}{2}(\rho(k) - \lambda), \frac{1}{2}(\rho(k) + \lambda); \alpha + 1; -\sinh^2 t\right).$$

Here ${}_2F_1$ is the Gauss hypergeometric function. For $\lambda \neq 1, 2, \dots$, there is another solution (1.15) of (1.23) on $(0, \infty)$ given by

$$(1.26) \quad \begin{aligned} \Phi_{-\sqrt{-1}\lambda}^{(\alpha, \beta)}(t) \\ = (2 \cosh t)^{\lambda - \rho(k)} {}_2F_1\left(\frac{1}{2}(\rho(k) - \lambda), \frac{1}{2}(\alpha - \beta + 1 - \lambda); 1 - \lambda; \cosh^{-2} t\right), \end{aligned}$$

which satisfies

$$(1.27) \quad \Phi_{-\sqrt{-1}\lambda}^{(\alpha, \beta)}(t) = e^{(\lambda - \rho)t}(1 + o(t)) \text{ as } t \rightarrow \infty.$$

For $\lambda \notin \mathbb{Z}$ we have

$$(1.28) \quad \varphi_{\sqrt{-1}\lambda}^{(\alpha, \beta)}(t) = c_{\alpha, \beta}(-\sqrt{-1}\lambda) \Phi_{-\sqrt{-1}\lambda}^{(\alpha, \beta)}(t) + c_{\alpha, \beta}(\sqrt{-1}\lambda) \Phi_{\sqrt{-1}\lambda}^{(\alpha, \beta)}(t),$$

where

$$(1.29) \quad c_{\alpha, \beta}(-\sqrt{-1}\lambda) = c(\lambda, k) = \frac{2^{\rho(k) - \lambda} \Gamma(\alpha + 1) \Gamma(\lambda)}{\Gamma(\frac{1}{2}(\lambda + \rho(k))) \Gamma(\frac{1}{2}(\lambda + \alpha - \beta + 1))}.$$

2. HYPERGEOMETRIC FUNCTION OF TYPE BC_n

2.1. Commuting family of differential operators. Let n be a positive integer greater than 1 and R be the root system of type BC_n

$$(2.1) \quad R_+ = \{e_p, 2e_p, e_i \pm e_j : 1 \leq p \leq n, 1 \leq i < j \leq n\},$$

where $\{e_1, \dots, e_n\}$ is the standard orthonormal basis of $E \simeq \mathbb{R}^n$. We call $\pm e_p$, $\pm(e_i \pm e_j)$, $\pm 2e_p$ short, middle, and long roots, respectively. We put

$$(2.2) \quad k_{e_p} = k_s, \quad k_{e_i + e_j} = k_m, \quad k_{2e_p} = k_l$$

for the multiplicities of short, middle, and long roots, respectively. Hereafter we assume that $k_m = 0$ or 1. Then the terms corresponding to the roots $e_i \pm e_j$ vanish in (1.6) and we have

$$(2.3) \quad \begin{aligned} \delta(k)^{\frac{1}{2}} \circ \{L(k) + (\rho(k), \rho(k))\} \circ \delta(k)^{-\frac{1}{2}} \\ = \sum_{j=1}^n \left(\partial_{e_j}^2 + \frac{k_s(1 - k_s - 2k_l)}{(e^{\frac{1}{2}e_j} - e^{-\frac{1}{2}e_j})^2} + \frac{4k_l(1 - k_l)}{(e^{e_j} - e^{-e_j})^2} \right). \end{aligned}$$

Let $t_j = e_j(\log a)/2$ ($j = 1, \dots, n$) be coordinates of $A \simeq \mathbb{R}^n$ and

$$(2.4) \quad a_t = \exp(\sum_{j=1}^n 2t_j e_j).$$

For $\lambda \in \mathfrak{h}^*$ put $\lambda_j = (\lambda, 2e_j)$. Then we have

$$(2.5) \quad \rho(k)_j = k_s + 2k_l + 2(n-j)k_m.$$

Let Δ_m be the Weyl denominator associated with middle roots

$$(2.6) \quad \begin{aligned} \Delta_m(a_t) &= \prod_{\alpha \in R_+, \text{middle roots}} (e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}) \\ &= 2^{\frac{1}{2}n(n-1)} \prod_{1 \leq i < j \leq n} (\cosh 2t_i - \cosh 2t_j). \end{aligned}$$

It is easy to see from (2.3) that

$$(2.7) \quad \Delta_m^{k_m} \circ (L(k) + (\rho(k), \rho(k))) \circ \Delta_m^{-k_m} = \sum_{j=1}^n L_j + n(k_s + 2k_l)^2,$$

where

$$(2.8) \quad L_j = \frac{\partial^2}{\partial t_j^2} + 2(k_s \coth t_j + 2k_l \coth 2t_j) \frac{\partial}{\partial t_j}.$$

Theorem 2.1. *If $k_m = 0$ or 1, then*

$$\mathbb{D}(k) = \{D_p = \Delta_m^{-k_m} \circ p(L_1, \dots, L_n) \circ \Delta_m^{k_m} : p \in \mathbb{R}[E]^W\}.$$

In particular, $\mathbb{D}(k)$ is generated by D_{p_j} ($j = 1, \dots, n$), where p_j is the j -th elementary symmetric function and $D_{p_1} = L(k) + (\rho(k), \rho(k))$.

Proof. Since L_1, \dots, L_n mutually commute and

$$\gamma(k)(\Delta_m^{-k_m} \circ L_j \circ \Delta_m^{k_m}) = \partial_{e_j}^2 - (k_s + 2k_l)^2,$$

the theorem follows from Theorem 1.1. □

Remark 2.2. The right hand side of (1.6) has the form of a Schrödinger operator and Theorem 1.1 tells that it defines a completely integrable system. Oshima [12] proved complete integrability of the Schrödinger operator

$$P = -\frac{1}{2} \sum_{j=1}^n \partial_{e_j}^2 + \sum_{1 \leq i < j \leq n} (u(t_i - t_j) + u(t_i + t_j)) + \sum_{1 \leq j \leq n} v(t_j)$$

with

$$\begin{aligned} u(x) &= C_1 \mathcal{P}(x) + C_2 \\ v(x) &= \frac{C_3 \mathcal{P}(x)^2 + C_4 \mathcal{P}(x)^3 + C_5 \mathcal{P}(x)^2 + C_6 \mathcal{P}(x) + C_7}{\mathcal{P}'(x)^2}. \end{aligned}$$

If $C_1 = 0$, then a result analogous to Theorem 2.1 holds.

Remark 2.3. If R is an arbitrary reduced root system and $k_\alpha = 0$ or 1 for all $\alpha \in R$, then the right hand side of (1.6) is just the Laplacian on the Euclidean space E . In this case, $\mathbb{D}(k)$ (taking conjugate by $\Delta_m^{k_m}$) consists of constant coefficient differential operators and the hypergeometric function is expressed by exponential functions. The case of all multiplicities equal to 1 is the case of complex semisimple Lie groups in the sense of Remark 1.3. Theorem 2.1 gives another case that $\mathbb{D}(k)$ has a simple simple expression.

2.2. The hypergeometric function. If $k_m = 0$ or 1 , then the Harish-Chandra series (1.15) is given by a product of the Harish-Chandra series' of the form (1.26) for the root system of $R = BC_1$.

Proposition 2.4. *Assume that $k_m = 0$ or 1 and let $\alpha = k_s + k_l - 1/2$, $\beta = k_l - 1/2$. If λ satisfies condition (1.13), then*

$$(2.9) \quad \Phi(\lambda, k; a) = \Delta_m(a_t)^{-k_m} \prod_{j=1}^n \Phi_{-\sqrt{-1}\lambda_j}^{(\alpha, \beta)}(t_j).$$

Proof. In view of (1.24), (1.26), (2.7), and (2.8), the right hand side of (2.9) is a solution of (1.14), where α and β are given by (1.21). We can see by elementary computations of power series that the right hand side of (2.9) has a series expansion of the form (1.15) as in the same way as the proof of [7, Theorem 1]. By the uniqueness of the Harish-Chandra series, (1.15) follows. \square

By virtue of Proposition 2.4, the hypergeometric function has a simple expression.

Theorem 2.5. *Let $\alpha = k_s + k_l - 1/2$, $\beta = k_l - 1/2$ and assume that $\alpha \neq 0, -1, -2, \dots$. If $k_m = 1$, then*

$$(2.10) \quad F(\lambda, k; a_t) = \frac{B}{\prod_{1 \leq i < j \leq n} (\lambda_i^2 - \lambda_j^2)} \cdot \frac{\det(\varphi_{\sqrt{-1}\lambda_i}^{(\alpha, \beta)}(t_j))_{1 \leq i, j \leq n}}{\Delta_m(a_t)},$$

where B is given by

$$(2.11) \quad B = (-1)^{\frac{1}{2}n(n-1)} 2^{2n(n-1)} \prod_{i=1}^{n-1} ((\alpha + i)^{n-i} i!).$$

If $k_m = 0$, then

$$(2.12) \quad F(\lambda, k; a_t) = \frac{1}{n!} \text{perm}(\varphi_{\sqrt{-1}\lambda_i}^{(\alpha, \beta)}(t_j))_{1 \leq i, j \leq n},$$

where $\text{perm}(M)$ denotes the permanent $\sum_{\sigma \in S_n} m_1 m_{\sigma(1)} \cdots m_n m_{\sigma(n)}$ of matrix $M = (m_{ij})_{1 \leq i, j \leq n}$.

Proof. First notice that the Weyl group of type BC_n is given by

$$W = \{w = (\varepsilon, \sigma) \in \{-1\}^n \times S_n : w(t_1, \dots, t_n) = (\varepsilon_1 t_{\sigma(1)}, \dots, \varepsilon_n t_{\sigma(n)})\}.$$

Assume that $k_m = 1$. The c -function for the middle roots (the product is take over the middle roots in (1.17)) is given by

$$(2.13) \quad \tilde{c}_m(\lambda, k) = \prod_{1 \leq i < j \leq n} \frac{\Gamma(\frac{1}{2}(\lambda_i + \lambda_j)) \Gamma(\frac{1}{2}(\lambda_i - \lambda_j))}{\Gamma(\frac{1}{2}(\lambda_i + \lambda_j) + 1) \Gamma(\frac{1}{2}(\lambda_i - \lambda_j) + 1)} \\ = \frac{2^{n(n-1)}}{\prod_{1 \leq i < j \leq n} (\lambda_i^2 - \lambda_j^2)}.$$

The c -function for e_j and $2e_j$ is given by

$$(2.14) \quad \tilde{c}_{e_j}(\lambda, k) \tilde{c}_{2e_j}(\lambda, k) = \frac{2^{-\lambda_j - k_s + 1} \Gamma(\lambda_j)}{\Gamma(\frac{1}{2}(\lambda_j + k_s + 1)) \Gamma(\frac{1}{2}(\lambda_j + k_s + 2k_l))} \\ = 2^{-2k_s - 2k_l + 1} \Gamma(k_s + k_l + \frac{1}{2})^{-1} c_{\alpha, \beta}(-\sqrt{-1}\lambda_j).$$

We have

$$(2.15) \quad \tilde{c}(\lambda, k) = \tilde{c}_m(\lambda, k) \prod_{j=1}^n \tilde{c}_{e_j}(\lambda, k) \tilde{c}_{2e_j}(\lambda, k) \\ = \frac{2^{n(n-2k_s-2k_l)}}{\Gamma(k_s + k_l + \frac{1}{2})^n \prod_{1 \leq i < j \leq n} (\lambda_i^2 - \lambda_j^2)} \prod_{j=1}^n c_{\alpha, \beta}(\lambda_j).$$

The hypergeometric function is given by

$$\Delta_m(a_t) F(\lambda, k; a_t) = \tilde{c}(\rho(k), k)^{-1} \sum_{w \in W} \tilde{c}(w\lambda, k) \Delta_m(a_t) \Phi(w\lambda, k, a_t) \\ = B \sum_{\sigma \in S_n, \varepsilon \in \{-1\}^n} \frac{1}{\prod_{i < j} (\lambda_{\sigma(i)}^2 - \lambda_{\sigma(j)}^2)} \prod_{l=1}^n c_{\alpha, \beta}(-\sqrt{-1}\varepsilon_l \lambda_{\sigma(l)}) \Phi_{-\sqrt{-1}\varepsilon_l \lambda_{\sigma(l)}}^{(\alpha, \beta)}(t_l) \\ = B \frac{1}{\prod_{i < j} (\lambda_i^2 - \lambda_j^2)} \sum_{\sigma \in S_n} \text{sgn } \sigma \prod_{l=1}^n \varphi_{\sqrt{-1}\lambda_{\sigma(l)}}^{(\alpha, \beta)}(t_l) \\ = B \frac{\det(\varphi_{\sqrt{-1}\lambda_i}^{(\alpha, \beta)}(t_j))_{i,j}}{\prod_{i < j} (\lambda_i^2 - \lambda_j^2)},$$

where

$$B = \frac{2^{n(n-1)}}{\tilde{c}(\rho(k), k) (2^{2k_s+2k_l-1} \Gamma(k_s + k_l + \frac{1}{2}))^n}.$$

The formula for B can be obtained by explicit computations.

Next suppose $k_m = 0$. Then $c_m(\lambda, k) = \lim_{k_m \rightarrow 0} \tilde{c}(\lambda, k) / \tilde{c}(\rho(k), k) = 1/n!$. Here $c_m(\lambda, k)$ is the c -function for the middle roots (the product is take over the middle roots in (1.18)). (2.12) follows by direct computation similar to that of deriving (2.10).

□

Remark 2.6. Let p and q ($p \leq q$) be positive integers and put $k_s = q - p$, $k_m = 1$, and $k_l = 1/2$. Then the hypergeometric function $F(\lambda, k; a_t)$ is the radial part of the

spherical function on $SU(p, q)/S(U(p) \times U(q))$. In this case Theorem 2.1, Theorem 2.4, and Theorem 2.1 were given by Berezin and Karpelevič [3] without proof and a complete proof was given by Hoogenboom [7].

We give two corollary of our results.

First we give a limit case of the hypergeometric function. We replace (t, λ) by $(\epsilon t, \epsilon^{-1}\lambda)$ and let $\epsilon \downarrow 0$. Then the hypergeometric equation (1.23) of type BC_1 becomes

$$(2.16) \quad \frac{d^2 u}{dt^2} + \frac{2\alpha + 1}{t} \frac{\partial^2 u}{\partial t^2} = \lambda^2 u.$$

Here we put $\alpha = k_s + k_l + 1/2$. There exists a unique even solution of (2.16) that is regular at 0 and $u(0) = 1$, which is given by

$$(2.17) \quad \mathcal{J}_\alpha(\sqrt{-1}\lambda t) = 2^\alpha \Gamma(\alpha + 1) (\sqrt{-1}\lambda t)^{-\alpha} J_\alpha(\sqrt{-1}\lambda t),$$

where J_α denote the usual Bessel function. Then it is known [8, §2.3] that

$$(2.18) \quad \lim_{\epsilon \downarrow 0} \varphi_{\sqrt{-1}\epsilon^{-1}\lambda}^{(\alpha, \beta)}(\epsilon t) = \mathcal{J}_\alpha(\sqrt{-1}\lambda t).$$

The limit of operator (1.5) become

$$(2.19) \quad L(k)^{\text{rat}} = \sum_{j=1}^n \partial_{\xi_j}^2 + \sum_{\alpha \in R_+} \frac{2k_\alpha}{\alpha} \partial_\alpha$$

and we have

$$(2.20) \quad \lim_{\epsilon \downarrow 0} \epsilon^{-n(n-1)} \Delta_m(a_{\epsilon t}) = \prod_{\alpha \in R_+, \text{middle roots}} \alpha(\log a_t).$$

We denote the right hand side of the above equation by $\Delta_{m, \text{rat}}(a_t)$. Put

$$(2.21) \quad L_j^{\text{rat}} = \frac{\partial^2}{\partial t_j^2} + \frac{2k_s + 2k_l + 2}{t_j} \frac{\partial}{\partial t_j}.$$

Then we have the following explicit expression of commuting family of differential operators including $L(k)^{\text{rat}}$.

Corollary 2.7. *If $k_m = 0$ or 1, then*

$$\{D_p^{\text{rat}} = \Delta_{m, \text{rat}}^{-k_m} \circ p(L_1, \dots, L_n) \circ \Delta_{m, \text{rat}}^{k_m} : p \in \mathbb{R}[E]^W\}$$

forms a commutative algebra of differential operators, which is generated by $\Delta_{m, \text{rat}}^{-k_m} \circ p_j(L_1, \dots, L_n) \circ \Delta_{m, \text{rat}}^{k_m}$, ($j = 1, \dots, n$), where p_j is the j -th elementary symmetric function. $D_{p_1}^{\text{rat}} = L(k)^{\text{rat}}$ and the principal symbol of $D_{p_j}^{\text{rat}}$ is p_j for $j = 1, \dots, n$.

By Theorem 2.5 and (2.18) we have the following limit formula.

Corollary 2.8. *Let $\alpha = k_s + k_l - 1/2$ and assume that $\alpha \neq 0, -1, -2, \dots$ and $\lambda_j \neq 0$, $t_j \neq 0$ ($j = 1, \dots, n$).*

If $k_m = 1$, then

$$(2.22) \quad \lim_{\epsilon \downarrow 0} F(\epsilon^{-1}\lambda, k; a_{\epsilon t}) = \frac{A}{\prod_{1 \leq i < j \leq n} (\lambda_i^2 - \lambda_j^2)} \cdot \frac{\det(\mathcal{J}_\alpha(\sqrt{-1}\lambda_i t_j))_{1 \leq i, j \leq n}}{\Delta_{m, \text{rat}}(a_t)},$$

where A is given by (2.11). If $k_m = 0$, then

$$(2.23) \quad \lim_{\epsilon \downarrow 0} F(\epsilon^{-1}\lambda, k; a_{\epsilon t}) = \frac{1}{n!} \text{perm}(\mathcal{J}_\alpha(\sqrt{-1}\lambda_i t_j))_{1 \leq i, j \leq n}.$$

Remark 2.9. In the group case that we mentioned in Remark 2.6, (2.22) was proved by Meaney [9]. It gives contraction of spherical functions between symmetric spaces of the non-compact type and the Euclidean type.

The right hand side of (2.22) and (2.23) give explicit expression for the Bessel function of type BC_n which was defined by Opdam [11, Definition 6.9]. The Bessel function of type BC_n for $k_m = 0$ or 1 is a W -invariant C^∞ joint-eigenfunction of the commuting family of differential operators given in Corollary 2.7 being equal to 1 at the origin.

The type of limit transition in Corollary 2.8 was given also by Ben Saïd and Ørsted [1, 2], and de Jeu [4].

Finally we give a formula for a Θ -spherical function. Let Ψ denote the set of simple roots in R_+ ,

$$\Psi = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_n\}.$$

For a subset $\Theta \subset \Psi$, let $\langle \Theta \rangle = R \cap \sum_{\alpha \in \Theta} \mathbb{Z}\alpha$ and define $\tilde{c}_\Theta(\lambda, k)$ by the product of the form (1.17) where the product is taken over $R_+ \cap \langle \Theta \rangle$ and let $c_\Theta(\lambda, k) = \tilde{c}_\Theta(\lambda, k) / \tilde{c}_\Theta(\rho(k), k)$.

We make a sum

$$(2.24) \quad F_\Theta(\lambda, k, a) = \sum_{w \in W_\Theta} c_\Theta(w\lambda, k) \Phi(w\lambda, k; a).$$

The sum of the form (2.24) is important in harmonic analysis of the spherical function on symmetric spaces (c.f. [10], [13, Chapter 6], [14]).

By Theorem 2.4, we can derive formulae for $F_\Theta(\lambda, k, a)$. For $\Theta = \Psi \setminus \{e_1 - e_2, \dots, e_{j-1} - e_j\}$ ($2 \leq j \leq n$) we have a formula for $F_\Theta(\lambda, k; a_t)$ that is similar to the formula for $F(\lambda, k; a_t)$ in Theorem 2.5.

If $\Theta = \{e_1 - e_2, \dots, e_{n-1} - e_n\}$, then $\langle \Theta \rangle$ is a root system of type A_{n-1} and we have the following result.

Corollary 2.10. *Assume that $k_m = 0$ or 1 and let $\Theta = \{e_1 - e_2, \dots, e_{n-1} - e_n\}$ and $\alpha = k_s + k_l - 1/2$, $\beta = k_l - 1/2$. Then $F_\Theta(\lambda, k; a_t)$ is holomorphic in λ in the region $\text{Re } \lambda_i > 0$ ($i = 1, \dots, n$). Moreover we have the following results.*

(i) Suppose $k_m = 1$ and put $\pi(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$. Then we have

$$(2.25) \quad F_\Theta(\lambda, k; a_t) = \frac{\pi(\rho(k))}{\pi(\lambda)} \cdot \frac{\det(\Phi_{\sqrt{-1}\lambda_i}^{(\alpha, \beta)}(t_j))_{1 \leq i, j \leq n}}{\Delta_m(a_t)}.$$

Moreover, if $\operatorname{Re} \lambda_i > 0$ ($i = 1, \dots, n$), then

$$(2.26) \quad \lim_{u \rightarrow \infty} e^{(\rho(k) - \lambda)(\log a_{(u, \dots, u)})} F_\Theta(\lambda, k; a_{(t_1+u, \dots, t_n+u)}) = \frac{\pi(\rho(k))}{\pi(\lambda)} \cdot \frac{\det(e^{\lambda_i t_j})_{1 \leq i, j \leq n}}{\pi(e^{2t_1}, \dots, e^{2t_n})}.$$

(ii) If $k_m = 0$, then

$$(2.27) \quad F_\Theta(\lambda, k; a_t) = \frac{1}{n!} \operatorname{perm}(\Phi_{\sqrt{-1}\lambda_i}^{(\alpha, \beta)}(t_j))_{1 \leq i, j \leq n}.$$

Moreover, if $\operatorname{Re} \lambda_i > 0$ ($i = 1, \dots, n$), then

$$(2.28) \quad \lim_{u \rightarrow \infty} e^{(\rho(k) - \lambda)(\log a_{(u, \dots, u)})} F_\Theta(\lambda, k; a_{(t_1+u, \dots, t_n+u)}) = \frac{1}{n!} \operatorname{perm}(e^{\lambda_i t_j})_{1 \leq i, j \leq n}.$$

Proof. $F_\Theta(\lambda, k; a_t)$ is holomorphic in the region $\operatorname{Re} \lambda_i > 0$ ($i = 1, \dots, n$) by [10, Theorem 8]. (2.25) and (2.27) follows by simple computations. (2.26) and (2.28) follows from (1.27). \square

Remark 2.11. (i) The right hand sides of (2.26) and (2.28) are hypergeometric function of type A_{n-1} with the multiplicity 1 and 0 respectively. Namely, the right hand side of (2.26) is the spherical function on $SL(n, \mathbb{C})/SU(n)$ (c.f. [6, Chapter IV Theorem 5.7]) and (2.28) is the normalized average of the exponential function $e^{(\lambda, t)}$ under the action of the symmetric group.

(ii) By [14, Proposition 2.6, Remark 6.13], the spherical function for a one-dimensional K -type $(\tau_{-\ell_1}, \tau_{-\ell_2})$ on $SU(p, q)$ can be written as the hypergeometric function $F(\lambda, k; a_t)$ with $k_s = m/2 - \ell_2$, $k_m = 1$, $k_l = 1/2 - \ell_1 - \ell_2$. Here $m = 1$ and $\ell_1 = \ell_2$ if $p \neq q$, and $m = 0$ if $p = q$. Thus spherical functions for one-dimensional K types on $SU(p, q)$ are given by Theorem 2.1. Conversely, by considering the universal covering group of $SU(p, q)$, we can take ℓ_1, ℓ_2 arbitrary complex numbers, hence the hypergeometric function (2.10) for any k_s and k_l corresponds to a spherical function on $SU(p, p)$.

By the above observation, the Plancherel formula for the integral transform with the kernel $F(\lambda, k; a)$ with $k_m = 1$ is a special case of [14, Theorem 6.11]. Notice that low dimensional spectra including discrete spectra appear in general. It seems to be possible to give an alternative proof of the Plancherel formula by rank one reduction as in [9, Theorem 22].

(iii) In Theorem 2.1 we give an explicit formula for the hypergeometric function of type BC_n with $k_m = 0, 1$ and k_s, k_l arbitrary. We obtain a formula of the hypergeometric function for $k_m \in \mathbb{Z}$ by applying Opdam's hypergeometric shift operator corresponding

to the middle roots, which is a differential operator of order $n(n-1)/2$ (c.f. [5, Definition 3.2.1]).

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